Continuous Function

$f: A \mapsto B$

$\mathrm{graph}(f) = \{(a, f(a)) : a \in A\} \subset A \times B$.

When $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, we can draw f on paper.

Def: $f: E \mapsto E'$

(E, d) and (E', d') are metric spaces.

Let $x_0 \in E$, we say that f is continuous at x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon.$$

We say that f is continuous if f is continuous at x_0, $\forall x_0 \in E$. $f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$

Example 1 ① Show that $f(x) = x^2$ is continuous.

Let $x_0 \in \mathbb{R}$, $\varepsilon > 0$.

$$d(f(x), f(x_0)) = |f(x) - f(x_0)|$$
$$= |x^2 - x_0^2|$$
$$= |x + x_0| \cdot |x - x_0|$$

If $x - x_0 < 1$, $\quad |x + x_0| \leq |x - x_0| + |2x_0|$

$$< 1 + 2|x_0|.$$

$$|f(x) - f(x_0)| < (1 + 2|x_0|) \cdot |x - x_0|$$

Let $\delta = \min\{\frac{\varepsilon}{1+2|x_0|}, 1\}$, then

if $|x - x_0| < \delta$, $|f(x) - f(x_0)| < (2|x_0|+1)\delta$

$$\leq \varepsilon$$

③ $f = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$ is not continuous at 0.

~~proof~~ Assume f is continuous.

Let $\varepsilon = 1/2$, Let $\delta > 0$. $f(\frac{\delta}{2}) = 1$.

$$|f(\tfrac{\delta}{2}) - f(0)| = 1 > \frac{1}{2} = \varepsilon$$

That means $\nexists\ \delta > 0$ such that

$$d(0, x) < \delta \Rightarrow d(f(0), f(x)) < \varepsilon.$$

③ $f: E \mapsto \mathbb{R}$, $a \in E$, $f(x) = d(a,x)$ is continuous.

proof Let $x_0 \in E$.

$$|f(x) - f(x_0)| = |d(x,a) - d(x_0,a)|$$
$$\leq d(x,x_0) < \delta = \varepsilon$$

then, given $\varepsilon > 0$, select $\delta = \varepsilon$.

Reminder $f^{-1}(A) = \{x : f(x) \in A\}$.

EX: $f(x) = x^2$, $f^{-1}([0,4]) = [-2,2]$.

Proposition $f: E \mapsto E'$, f is continuous $\Leftrightarrow$ $f^{-1}(U')$ is open in E for all U' open in E'.

proof: $\Rightarrow$) Assume f is continuous.

Let $U' \subset E$ with U' open. We need to show that $f^{-1}(U')$ is open in E.

Let $x_0 \in f^{-1}(U')$, we need to show that $\exists\, \delta > 0$, s.t. $d(x, x_0) < \delta \Rightarrow x \in f^{-1}(U')$.

Since $x_0 \in f^{-1}(U')$, $f(x_0) \in U'$.

Since U' is open, $f(x_0) \subset U'$.

$\exists\, \varepsilon > 0$, s.t. $d(y, f(x_0)) < \varepsilon \Rightarrow y \in U'$.

Since f is continuous at x_0, $\exists\, \delta > 0$, s.t.

$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)) \subset U'$

Then $B_\delta(x_0) \subset f^{-1}(U')$.

$\Leftarrow$) Let $x_0 \in E$, $\varepsilon > 0$. I need to find $\delta > 0$ s.t. $f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$

Since $B_\varepsilon(f(x_0))$ is open, so is $f^{-1}(B_\varepsilon(f(x_0)))$.

since $x_0 \in f^{-1}(B_\varepsilon(f(x_0)))$ and $f^{-1}(B_\varepsilon(f(x_0)))$

$\exists\, \delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x)))$.

Thus, $f(B_\delta(x_0)) \subset B_\varepsilon(f(x))$

Thus, f is continuous at x_0.

Proposition Let $f : E \mapsto E'$, $g : E' \mapsto E''$
Assume f is continuous at x_0 and g is continuous at $f(x_0)$. Then $g \circ f : E \mapsto E''$ is continuous at x_0.

proof Let $\delta > 0$. Since g is continuous at $f(x_0)$
$\exists\, \alpha > 0$, s.t.

$$d(y, f(x_0)) < \alpha \Rightarrow d(g(y), g(f(x_0))) < \varepsilon$$

Since f is continuous at x_0, $\exists\, \delta > 0$, s.t.

$$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \alpha.$$

Thus, $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \alpha$

$$\Rightarrow d(g\circ f(x), g \circ f(x_0)) < \varepsilon.$$

Thus, $g \circ f$ is continuous at x_0.

Practice E is complete and totally bounded
$\Rightarrow E$ is compact.

def: E is totally bounded if
$\forall \varepsilon > 0$, $\exists\, n \in \mathbb{N}$, and $x_1, \cdots, x_n \in E$.

s.t. $E = \bigcup \{x : d(x, x_i) \leq \varepsilon\}$.

note: totally bounded $\Rightarrow$ bounded in $\mathbb{R}$.

Limits:

Def: $x_0 \in E$, $f : E \mapsto E'$ or $f : E - \{x_0\} \mapsto E'$

We say $\lim_{x \to x_0} f(x) = q$ if $\bar{f} : E \mapsto E'$

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \neq x_0 \\ q, & \text{if } x = x_0 \end{cases}$$

is continuous at x_0.

Obs $\lim_{x \to x_0} f(x) = q \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t.
$d(x, x_0) < \delta \Rightarrow d(f(x), q) < \varepsilon$.